MTH 295
Fall 2019
Homework 7
Due Thursday, 10/31 Name: Key

1) Find the Laplace transform of each of the following using the attached table.

a) $f(t)=e^{-2t}\sin(5t)$

$$F(s)=\boxed{\frac{5}{(s+2)^2+25}}$$ #19 (or #7 and #29)

b) $f(t)=te^{-t}\cos 2t$

$$F(s)=\boxed{\frac{(s+1)^2-4}{\{(s+1)^2+4\}^2}}$$ #10 and #29

c) $f(t)=t^5e^{-\pi t}$

$$F(s)=\boxed{\frac{5!}{(s+\pi)^6}}$$ #23

d) $f(t)=5\sin 3t-17e^{-2t}$

$$F(s)=5\mathcal{L}\{\sin 3t\}-17\mathcal{L}\{e^{-2t}\}$$ by linearity of $\mathcal{L}$

$$=5\cdot\frac{3}{s^2+9}-17\cdot\frac{1}{s+2}$$ #7 and #2

$$=\boxed{\frac{15}{s^2+9}-\frac{17}{s+2}}$$

2) Find the inverse Laplace transform of the following using the attached table.

a) $F(s)=\dfrac{s}{(s-2)^2+11}$ $= \dfrac{s-2+2}{(s-2)^2+11}$

$= \dfrac{s-2}{(s-2)^2+11} + \dfrac{2}{(s-2)^2+11}$

$= \dfrac{s-2}{(s-2)^2+(\sqrt{11})^2} + \dfrac{2}{\sqrt{11}} \cdot \dfrac{\sqrt{11}}{(s-2)^2+(\sqrt{11})^2}$

$f(t) = \boxed{e^{2t}\cos(\sqrt{11}\,t) + \frac{2}{\sqrt{11}} e^{2t}\sin(\sqrt{11}\,t)}$ #19 and #20

b) $F(s)=\dfrac{s+1}{s^2-9}$ $= \dfrac{s+1}{(s+3)(s-3)}$

$= \frac{1}{3}\cdot\frac{1}{s+3} + \frac{2}{3}\cdot\frac{1}{s-3}$

so $f(t) = \boxed{\frac{1}{3}e^{-3t} + \frac{2}{3}e^{3t}}$ #2

$\dfrac{s+1}{(s+3)(s-3)} = \dfrac{A}{s+3} + \dfrac{B}{s-3}$

$s+1 = As-3A+Bs+3B$

$A+B=1$

$-A+B=1/3$

$2B = 4/3$

$B=2/3$

$A = 1-2/3 = \frac{1}{3}$

or $F(s) = \dfrac{s}{s^2-9} + \dfrac{1}{s^2-9}$

$\boxed{f(t) = \cosh(3t) + \frac{1}{3}\sin(3t)}$ #21 and #22

you show these are same.

c) $F(s)=\dfrac{1}{(s+1)(s^2+1)}$

$= \frac{1}{2}\cdot\frac{1}{s+1} - \frac{1}{2}\frac{s}{s^2+1} + \frac{1}{2}\frac{1}{s^2+1}$

$f(t) = \boxed{\frac{1}{2}e^{-t} - \frac{1}{2}\cos t + \frac{1}{2}\sin t}$

#2, #7, #8

$\dfrac{1}{(s+1)(s^2+1)} = \dfrac{A}{s+1} + \dfrac{Bs+C}{s^2+1}$

$1 = As^2+A+Bs^2+Bs+Cs+C$

$A+B=0$

$B+C=0$

$A+C=1$

$A-C=0$

$2A=1$

$A=1/2$

$B=-1/2$

$C=1/2$

3) Solve the IVP $y''-2y'+y=3e^t$, $y(0)=y'(0)=1$ using the method of Laplace transforms.

$$s^2Y(s)-sy(0)-y'(0)-2\{sY(s)-y(0)\}+Y(s)=3\frac{1}{s-1} \qquad (\#37, \#2)$$

$$s^2Y(s)-s-1-2sY(s)+2+Y(s)=3\cdot\frac{1}{s-1} \qquad \text{(apply conditions)}$$

$$(s^2-2s+1)Y(s)=3\cdot\frac{1}{s-1}+s-1$$

$$(s-1)^2Y(s)=\frac{3}{s-1}+s-1$$

$$Y(s)=\frac{3}{(s-1)^3}+\frac{1}{s-1}$$

$$=\frac{3}{2}\cdot\frac{2}{(s-1)^3}+\frac{1}{s-1}$$

$$y(t)=\mathcal{L}^{-1}\left\{\frac{3}{2}\cdot\frac{2}{(s-1)^3}+\frac{1}{s-1}\right\}$$

$$=\frac{3}{2}\mathcal{L}^{-1}\left\{\frac{2}{(s-1)^3}\right\}+\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} \qquad \text{(by linearity)}$$

$$=\frac{3}{2}\cdot t^2e^t+e^t \qquad (\#23 \text{ and } \#2)$$

$$\boxed{y(t)=\frac{3}{2}t^2e^t+e^t}$$

4) Solve the IVP $y''' - 6y'' + 11y' - 6y = e^{4t}, y(0) = y'(0) = y''(0) = 0$ using the method of Laplace transforms.

$$s^3Y(s) - s^2y(0) - sy'(0) - y''(0) - 6\{s^2Y(s) - sy(0) - y'(0)\} + 11\{sY(s) - y(0)\} - 6Y(s) = \frac{1}{s-4}$$

$$s^3Y(s) - 0 - 0 - 0 - 6\{s^2Y(s) - 0 - 0\} + 11\{sY(s) - 0\} - 6Y(s) = \frac{1}{s-4}$$

$$\underbrace{(s^3 - 6s^2 + 11s - 6)}_{\text{this factors}}Y(s) = \frac{1}{s-4}$$

$$\begin{array}{r|rrrr} 1 & 1 & -6 & 11 & -6 \\ & & 1 & -5 & 6 \\ \hline & 1 & -5 & 6 & 0 \end{array}$$

$$(s-1)(s^2 - 5s + 6)Y(s) = \frac{1}{s-4}$$

$$(s-1)(s-2)(s-3)Y(s) = \frac{1}{s-4}$$

$$Y(s) = \frac{1}{(s-1)(s-2)(s-3)(s-4)}$$

Then by partial fractions –

$$\frac{1}{(s-1)(s-2)(s-3)(s-4)} = \frac{A}{s-1} + \frac{B}{s-2} + \frac{C}{s-3} + \frac{D}{s-4}$$

$$1 = A(s-2)(s-3)(s-4) + B(s-1)(s-3)(s-4) + C(s-1)(s-2)(s-4) + D(s-1)(s-2)(s-3)$$

so if $s = 1$ –

$-6A = 1$

$A = -1/6$

Similarly, letting $s = 2, 3, 4$

$B = 1/2$

$C = -1/2$

$D = 1/6$

so $Y(s) = -\frac{1}{6}\cdot\frac{1}{s-1} + \frac{1}{2}\cdot\frac{1}{s-2} - \frac{1}{2}\cdot\frac{1}{s-3} + \frac{1}{6}\cdot\frac{1}{s-4}$

and $y(t) = \boxed{-\frac{1}{6}e^{t} + \frac{1}{2}e^{2t} - \frac{1}{2}e^{3t} + \frac{1}{6}e^{4t}}$ (from #2)

Table of Laplace Transforms

	$f(t)=\mathcal{L}^{-1}\{F(s)\}$	$F(s)=\mathcal{L}\{f(t)\}$		$f(t)=\mathcal{L}^{-1}\{F(s)\}$	$F(s)=\mathcal{L}\{f(t)\}$
1.	1	$\frac{1}{s}$	2.	$\mathbf{e}^{at}$	$\frac{1}{s-a}$
3.	$t^n,\quad n=1,2,3,\ldots$	$\frac{n!}{s^{n+1}}$	4.	$t^p, p>-1$	$\frac{\Gamma(p+1)}{s^{p+1}}$
5.	$\sqrt{t}$	$\frac{\sqrt{\pi}}{2s^{\frac{3}{2}}}$	6.	$t^{n-\frac{1}{2}},\quad n=1,2,3,\ldots$	$\frac{1\cdot3\cdot5\cdots(2n-1)\sqrt{\pi}}{2^n s^{n+\frac{1}{2}}}$
7.	$\sin(at)$	$\frac{a}{s^2+a^2}$	8.	$\cos(at)$	$\frac{s}{s^2+a^2}$
9.	$t\sin(at)$	$\frac{2as}{(s^2+a^2)^2}$	10.	$t\cos(at)$	$\frac{s^2-a^2}{(s^2+a^2)^2}$
11.	$\sin(at)-at\cos(at)$	$\frac{2a^3}{(s^2+a^2)^2}$	12.	$\sin(at)+at\cos(at)$	$\frac{2as^2}{(s^2+a^2)^2}$
13.	$\cos(at)-at\sin(at)$	$\frac{s(s^2-a^2)}{(s^2+a^2)^2}$	14.	$\cos(at)+at\sin(at)$	$\frac{s(s^2+3a^2)}{(s^2+a^2)^2}$
15.	$\sin(at+b)$	$\frac{s\sin(b)+a\cos(b)}{s^2+a^2}$	16.	$\cos(at+b)$	$\frac{s\cos(b)-a\sin(b)}{s^2+a^2}$
17.	$\sinh(at)$	$\frac{a}{s^2-a^2}$	18.	$\cosh(at)$	$\frac{s}{s^2-a^2}$
19.	$\mathbf{e}^{at}\sin(bt)$	$\frac{b}{(s-a)^2+b^2}$	20.	$\mathbf{e}^{at}\cos(bt)$	$\frac{s-a}{(s-a)^2+b^2}$
21.	$\mathbf{e}^{at}\sinh(bt)$	$\frac{b}{(s-a)^2-b^2}$	22.	$\mathbf{e}^{at}\cosh(bt)$	$\frac{s-a}{(s-a)^2-b^2}$
23.	$t^n\mathbf{e}^{at},\quad n=1,2,3,\ldots$	$\frac{n!}{(s-a)^{n+1}}$	24.	$f(ct)$	$\frac{1}{c}F\left(\frac{s}{c}\right)$
25.	$u_c(t)=u(t-c)$ Heaviside Function	$\frac{\mathbf{e}^{-cs}}{s}$	26.	$\delta(t-c)$ Dirac Delta Function	$\mathbf{e}^{-cs}$
27.	$u_c(t)f(t-c)$	$\mathbf{e}^{-cs}F(s)$	28.	$u_c(t)g(t)$	$\mathbf{e}^{-cs}\mathcal{L}\{g(t+c)\}$
29.	$\mathbf{e}^{ct}f(t)$	$F(s-c)$	30.	$t^n f(t),\quad n=1,2,3,\ldots$	$(-1)^n F^{(n)}(s)$
31.	$\frac{1}{t}f(t)$	$\int_s^\infty F(u)\,du$	32.	$\int_0^t f(v)\,dv$	$\frac{F(s)}{s}$
33.	$\int_0^t f(t-\tau)g(\tau)\,d\tau$	$F(s)G(s)$	34.	$f(t+T)=f(t)$	$\frac{\int_0^T \mathbf{e}^{-st}f(t)\,dt}{1-\mathbf{e}^{-sT}}$
35.	$f'(t)$	$sF(s)-f(0)$	36.	$f''(t)$	$s^2F(s)-sf(0)-f'(0)$
37.	$f^{(n)}(t)$	$s^nF(s)-s^{n-1}f(0)-s^{n-2}f'(0)\cdots-sf^{(n-2)}(0)-f^{(n-1)}(0)$			